

On a geometric black hole of a compact manifold

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Abstract: Using a smooth triangulation and a Riemannian metric on a compact, connected, closed manifold M^n of dimension n we claim that every such M^n can be represented as a union of a n -dimensional cell C^n and a connected union K^{n-1} ($\dim K^{n-1} \leq n - 1$) of some finite number of subsimplexes of the triangulation. A sufficiently small closed neighborhood of K^{n-1} is called a *geometric black hole*. Any smooth tensor field K (a fiber bundle) can be deformed into a continuous and sectionally smooth tensor field \bar{K} where \bar{K} has a very simple construction out of the black hole.

Keywords: Compact manifolds, Riemannian metric, triangulation, homotopy, deformations of tensor fields and fiber bundles.

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0. About the term of a geometric black hole (GBH)

The following variants of the term were considered for *GBH*: topological black hole, topologically-geometric black hole, generalized black hole, geometric black object *etc.* We have to underline that every word of the term is important to distinguish a *GBH* from the famous black holes in the theory of relativity (further, classical black holes or *CBH*).

I had the following reasons to use the term above for a *GBH*.

- 1) A *GBH* exists in any smooth compact manifold M and determines topological and sectionally-smooth properties of M .
- 2) Any smooth tensor field K (for example a pseudoRiemannian or Finsler metric) or other structure can be deformed towards a *GBH* into a continuous and sectionally-smooth tensor field or structure where a parameter t of the deformation can be considered as a time.
- 3) A behavior of a *GBH* is very familiar to that of a *CBH*.
Hypothesis: if we consider a *CBH* defined by a pseudoRiemannian (Finsler) metric on a compact manifold M then after some deformation of the metric towards the *GBH* the *CBH* has to belong the *GBH*.
- 4) *Example.* If we consider a sphere then a *GBH* is simply a small black closed ball in the sphere. After deformations of all the tensor fields towards the *GBH* we have obtained a situation reminding us that of the «big explosion».
- 5) I have used the term *GBH* in abstracts of several conferences on topology and geometry of manifolds.

Thus, I think that the object *GBH* could be useful for researchers in mathematics and theoretical physics.

1. On extension of coordinate neighborhood

1°. Let M^n be a connected, compact, closed and smooth manifold of dimension n and C^n be a cell (coordinate neighborhood) on M^n . We can fix some Riemannian metric g on the manifold M^n which defines the length of arc of a piecewise smooth curve and the continuous function $\rho(x; y)$ of the distance between two points $x, y \in M^n$. The topology defined by the function of distance (metric) ρ is the same as the topology of the manifold M^n , [3].

A standard simplex Δ^n of dimension n is the set of points $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$ defined by conditions

$$0 \leq x_i \leq 1, \quad i=1, \dots, n, \quad x_1 + x_2 + \dots + x_n \leq 1.$$

We consider the interval of a straight line connected the center of some face of Δ^n and the vertex which is opposite to this face. It is clear that the center of Δ^n belongs to the interval. We can decompose Δ^n as a set of intervals which are parallel to that mentioned above. If the center of Δ^n is connected by intervals with points of some face of Δ^n then a subsimplex of Δ^n is obtained. All the faces of Δ^n considered, Δ^n is seen as a set of all such subsimplexes. Let $U(\Delta^n)$ be some open neighborhood of Δ^n in \mathbf{R}^n . A diffeomorphism $\varphi : U(\Delta^n) \rightarrow M^n$ ($\delta^n = \varphi(\Delta^n)$) is called a singular n -simplex on the manifold M^n . *Faces, edges, the center, vertexes* of the simplex δ^n are defined as the images of those of Δ^n with respect to φ .

The manifold M^n is triangulable, [7]. It means that for any $l, 0 \leq l \leq n$ such a finite set Φ^l of diffeomorphisms $\varphi : \Delta^l \rightarrow M^n$ is defined that

- a) M^n is a disjunct union of images $\varphi(Int \Delta^l)$, $\varphi \in \Phi^l$;
- b) if $\varphi \in \Phi^l$ then $\varphi \circ \varepsilon_i \in \Phi^{l-1}$ for every i where $\varepsilon_i : \Delta^{l-1} \rightarrow \Delta^l$ is the linear mapping transferring the vertexes v_0, \dots, v_{k-1} of the simplex Δ^{l-1} in the vertexes v_0, \dots, v_k of the simplex Δ^l .

2°. Let δ_0^n be some simplex of the fixed triangulation of the manifold M^n . We paint the inner part $Int \delta_0^n$ of the simplex δ_0^n in white and the boundary $\partial \delta_0^n$ of δ_0^n in black. There exist coordinates on $Int \delta_0^n$ given by diffeomorphism φ_0 . A subsimplex $\delta_{01}^{n-1} \subset \delta_0^n$ is defined by a black face $\delta_{01}^{n-1} \subset \delta_0^n$ and the center c_0 of δ_0^n . We connect c_0 with the center d_0 of the face δ_{01}^{n-1} and decompose the subsimplex δ_{01}^n as a set of intervals which are parallel to the interval $c_0 d_0$. The face δ_{01}^{n-1} is a face of some simplex δ_1^n that has not been painted. We draw an interval

between d_0 and the vertex v_1 of the subsimplex δ_1^n which is opposite to the face δ_{01}^{n-1} then we decompose δ_1^n as a set of intervals which are parallel to the interval $d_0 v_1$. The set $\delta_{01}^n \cup \delta_1^n$ is a union of such broken lines every one from which consists of two intervals where the endpoint of the first interval coincides with the beginning of the second interval (in the face δ_{01}^{n-1}) the first interval belongs to δ_{01}^n and the second interval belongs to δ_1^n . We construct a homeomorphism (extension) $\varphi_{01}^1: \text{Int}\delta_{01}^n \rightarrow \text{Int}(\delta_{01}^n \cup \delta_1^n)$. Let us consider a point $x \in \text{Int}\delta_{01}^n$ and let x belong to a broken line consisting of two intervals the first interval is of a length of s_1 and the second interval is of a length of s_2 and let x be at a distance of s from the beginning of the first interval. Then we suppose that $\varphi_{01}^1(x)$ belongs to the same broken line at a distance of $\frac{s_1 + s_2}{s_1} \cdot s$ from the beginning of the first interval. It is clear that

φ_{01}^1 is a homeomorphism giving coordinates on $\text{Int}(\delta_{01}^n \cup \delta_1^n)$. We paint points of $\text{Int}(\delta_{01}^n \cup \delta_1^n)$ white. Assuming the coordinates of points of white initial faces of subsimplex δ_{01}^n to be fixed we obtain correctly introduced coordinates on $\text{Int}(\delta_{01}^n \cup \delta_1^n)$. The set $\sigma_1 = \delta_{01}^n \cup \delta_1^n$ is called a *canonical polyhedron*. We paint faces of the boundary $\partial\sigma_1$ black.

We describe the contents of the successive step of the algorithm of extension of coordinate neighborhood. Let us have a canonical polyhedron σ_{k-1} with white inner points (they have introduced *white coordinates*) and the black boundary $\partial\sigma_{k-1}$. We look for such an n -simplex in σ_{k-1} , let it be δ_0^n that has such a black face, let it be δ_{01}^{n-1} that is simultaneously a face of some n -simplex, let it be δ_1^n , inner points of which are not painted. Then we apply the procedure described above to the pair δ_0^n, δ_1^n . As a result we have a polyhedron σ_k with one simplex more than σ_{k-1} has. Points of $\text{Int}\sigma_k$ are painted in white and the boundary $\partial\sigma_k$ is painted in black. The process is finished in the case when all the black faces of the last polyhedron border on the set of white points (the cell) from two sides.

After that all the points of the manifold M^n are painted in black or white, otherwise we would have that $M^n = M_0^n \cup M_1^n$ (the points of M_0^n would be painted and those of M_1^n would be not) with M_0^n and M_1^n being unconnected, which would contradict of connectivity of M^n .

Thus, we have proved the following

Theorem 1. *Let M^n be a connected, compact, closed, smooth manifold of dimension n . Then $M^n = C^n \cup K^{n-1}$, $C^n \cap K^{n-1} = \emptyset$, where C^n is an n -dimensional cell and K^{n-1} is a union of some finite number of $(n-1)$ -simplexes of the triangulation.*

3°. We consider the initial simplex δ_0^n of the triangulation and its center c_0 . Drawing intervals between the point c_0 and points of all the faces of δ_0^n we obtain a decomposition of δ_0^n as a set of the intervals. In 2° the homeomorphism $\Psi: \text{Int}\delta_0^n \rightarrow C^n$ was constructed and Ψ evidently maps every interval above on a piecewise smooth broken line γ in C^n . We denote $\tilde{M}^n = M^n \setminus \{c_0\}$. \tilde{M}^n is a connected and simply connected manifold if M^n is that. Let $I=[0;1]$, we define a homotopy $F: \tilde{M}^n \times I \rightarrow \tilde{M}^n: (x; t) \mapsto y=F(x;t)$ in the following way

- a) $F(z; t)=z$ for every point $z \in K^{n-1}$;
- b) if a point x belongs to the broken line γ in C^n and the distance between x and its limit point $z \in K^{n-1}$ is $s(x)$ then $y=F(x; t)$ is on the same broken line γ at a distance of $(1-t)s(x)$ from the point z .

It is clear that $F(x;0)=x$, $F(x;1)=z$ and we have obtained the following

Theorem 2. *The spaces \tilde{M}^n and K^{n-1} are homotopy-equivalent, in particular, the groups of singular homologies $H_k(\tilde{M}^n)$ and $H_k(K^{n-1})$ are isomorphic for every k .*

Corollary 2.1. *The space K^{n-1} is connected and if M^n is simply connected then K^{n-1} is simply connected too.*

Remark. *The white coordinates are extended from the simplex δ_0^n in the simplex δ_1^n through the face δ_{01}^{n-1} hence $\text{Int}\delta_{01}^{n-1}$ has also the white coordinates. On the other hand there exist two linear structures (intervals, the center etc) on δ_{01}^n induced from δ_0^n and δ_1^n respectively. Further, we set that the linear structure of δ_{01}^{n-1} is the structure induced from δ_0^n .*

2. Deformation of a tensor field and a fibre bundle towards a geometric black hole of a compact manifold

1°. Let $L(M^n)$ and $L(C^n)$ be the principal fibre bundles of linear frames of the manifolds M^n and C^n . The diffeomorphism φ_0 (where $\delta_0^n = \varphi_0(\Delta^n)$) defines the coordinates (x_1, \dots, x_n) in some neighborhood of the simplex δ_0^n and the corresponding vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ on this neighborhood (a local cross-section of $L(M^n)$). Similarly, the diffeomorphism φ_1 (where $\delta_1^n = \varphi_1(\Delta^n)$) defines the coordinates (y_1, \dots, y_n) in some neighborhood of the simplex δ_1^n and the vector fields $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ on this neighborhood. We have assumed that the white face δ_{01}^{n-1}

has the equation: $y_1=0$ (it can always be obtained by corresponding linear change of variables in $\mathbf{R}^n \supset \Delta^n$). The vector fields $X_i = \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}, \quad i, j = \overline{1, n}$, are defined on

the face δ_{01}^{n-1} therefore for any point $x \in \delta_{01}^{n-1}$ we have $X_i = \sum_{j=1}^n f_{ij}(x) \frac{\partial}{\partial y_j}$ where the

functions $f_{ij}(x)$ are smooth. We decompose δ_1^n as a union of the intervals having the following equations: $y_1=t, y_2=c_2, y_3=c_3, \dots, y_n=c_n$, where $0, c_2, \dots, c_n$ are the coordinates of the beginning y_0 of the corresponding interval. For any point $y \in \delta_1^n$ we assume $f_{ij}(y) = f_{ij}(y_0)$ where $y_0 \in \delta_{01}^{n-1}$ is the beginning of the interval where the point y is situated. The vector fields $X_i, \quad i = \overline{1, n}$, are defined on δ_1^n by the formula $X_i = \sum_{j=1}^n f_{ij}(y) \frac{\partial}{\partial y_j}$. It is obvious that the constructed vector fields

$X_i, \quad i = \overline{1, n}$, are continuous on $\delta_0^n \cup \delta_1^n$ and smooth in any point $x \in \delta_0^n \cup \delta_1^n, x \notin \delta_{01}^{n-1}$.

For the process of the extension of a coordinate neighborhood $(1, 2^\circ)$ we can consider the process of the extension of the vector fields X_1, \dots, X_n . If these fields are defined on a polyhedron σ_{k-1} and in order to get a polyhedron σ_k we use simplexes δ_0^n, δ_1^n then we apply the procedure described above to obtain the vector fields on σ_k . As a result we obtain correctly defined vector fields X_1, \dots, X_n on C^n i.e. a cross-section of $L(C^n)$.

So, we come to the following

Proposition 3. *Let $M^n = C^n \cup K^{n-1}$ be the decomposition from the theorem 1. Then there exists a continuous cross-section of $L(C^n)$: $x \rightarrow (X_1, \dots, X_n)_x, x \in C^n$. If a point $x \in C^n$ does not belong to the subsimplexes of the triangulation then the cross-section above is smooth at the point x .*

We consider a tensor of type (r, s) on \mathbf{R}^n :

$$K^0 = \sum k_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(0) e_{\lambda_1} \otimes \dots \otimes e_{\lambda_r} \otimes e^{\mu_1} \otimes \dots \otimes e^{\mu_s},$$

where e_1, \dots, e_n is the standard basis of \mathbf{R}^n and e^1, \dots, e^n is the dual basis of \mathbf{R}^{n*} .

A tensor field of type (r, s) is defined on C^n :

$$K^0 = \sum k_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(0) X_{\lambda_1} \otimes \dots \otimes X_{\lambda_r} \otimes X^{\mu_1} \otimes \dots \otimes X^{\mu_s} \quad (1)$$

Since the functions $k_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}$ are constant on C^n we obtain that the tensor field K^0 is O -deformable on C^n i.e. some G -structure on C^n is defined by K^0 (see [1], [5]). If the cross-section $(X_1, \dots, X_n)_x$ is smooth at a point $x \in C^n$ then the tensor field K^0 is also smooth at the point.

2°. For any point $z \in K^{n-1}$ we can consider the closed geodesic ball $\overline{B}(z, \varepsilon)$ of a small radius $\varepsilon > 0$. Let $Tb(K^{n-1}, \varepsilon) = \bigcup_{z \in K^{n-1}} \overline{B}(z, \varepsilon) = GBH(\varepsilon)$.

Definition 1. We call the set $GBH(\varepsilon)$ a geometric black hole of radius $\varepsilon > 0$ of the manifold M^n if $M^n \setminus GBH(\varepsilon)$ is a cell (it is true for some small ε). We paint the points of $GBH(\varepsilon)$ in black.

Any piecewise smooth broken line γ considered in **1**, **3°** can be represented as $\gamma = \gamma_0 \cup \gamma_1$ where $\gamma_1 = \gamma \cap GBH(\varepsilon)$, $\gamma_0 = \gamma \setminus \gamma_1$. The points of γ_0 are painted in white and the points of γ_1 are painted in black. Let the segment γ_0 have a length s_0 and the segment γ_1 have a length s_1 then $(s_0 + s_1)$ is a length of the broken line γ from c_0 to $z \in K^{n-1}$.

Let $K(x)$, $x \in M^n$, be a tensor field of type (r, s) and $K^0 = K(c_0)$ where c_0 is the center of the initial simplex δ_0^n of the triangulation of M^n . Deformations of structures was considered in [2]. So, we shall construct a deformation $\bar{K}(x)$ of the tensor field $K(x)$ on the manifold M^n .

- 1) If a point $z \in K^{n-1}$ then $\bar{K}(z) = K(z)$.
- 2) If a point $x \in M^n \setminus GBH(\varepsilon)$ then $\bar{K} = K^0 = K(c_0)$ where K^0 is defined by the formula (1).
- 3) We assume that $K(x) = \sum k_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(x) X_1 \otimes \dots \otimes X_{\lambda_r} \otimes X^{\mu_1} \otimes \dots \otimes X^{\mu_s}$, $x \in C^n$, where X_1, \dots, X_r are the vector fields from the proposition 3, a point x belongs a broken line γ and $s(x)$ is the distance from x to c_0 along the broken line γ . For any point $y \in \gamma_1$ we define the tensor field

$$\bar{K}(y) = \sum \bar{k}_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(y) X_1 \otimes \dots \otimes X_{\lambda_r} \otimes X^{\mu_1} \otimes \dots \otimes X^{\mu_s}$$

in the following way: $\bar{k}_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(y) = k_{\mu_1 \dots \mu_s}^{\lambda_1 \dots \lambda_r}(x)$ where $s(x) = \frac{s(y) - s_0}{s_1}(s_0 + s_1)$, $s(y)$ is

the distance from y to c_0 along the broken line γ .

It is easy to see that the constructed tensor field \bar{K} is continuous and sectionally smooth, \bar{K} is not smooth on the boundary of $GBH(\varepsilon)$ and in the points of C^n where the cross-section $(X_1, \dots, X_n)_x$ is not smooth.

Let L be some operator defined on the algebra (or some subalgebra) of all the tensor fields on the manifold M^n and $L(K) = K_1$ for a tensor field K .

Definition 2. An operator \bar{L} is called a deformation of L towards $GBH(\varepsilon)$ if it is defined by condition $\bar{L}(K) = \bar{K}_1$.

3°. Further, we follow [4], [6].

A fiber bundle (E, π, M^n, F) consists of manifolds (spaces) E, M^n, F and a smooth (continuous) mapping $\pi: E \rightarrow M^n$, furthermore each $x \in M^n$ has an open neighborhood U such that $E|_U \cong \pi^{-1}(U)$ is diffeomorphic (homeomorphic) to $U \times F$ via a fiber respecting diffeomorphism (homeomorphism):

$$\begin{array}{ccc}
E|_U & \xleftrightarrow{\psi} & U \times F \\
\pi \downarrow & & \downarrow pr_1 \\
U & \xleftrightarrow{\quad} & U
\end{array}$$

E is called the total space, M^n is called the base space, π is called the projection, F is called standard fiber, (U, ψ) is called a fiber chart.

A collection of fiber charts (U_α, ψ_α) , such that $\{U_\alpha\}$ is an open cover of M^n , is called a fiber bundle atlas. If we fix such an atlas, then $\psi_\alpha \circ \psi_\beta^{-1}(x, a) = (x, \psi_{\alpha\beta}(x, a))$, where $\psi_{\alpha\beta} : (U_\alpha \times U_\beta) \times F \rightarrow F$ is smooth (continuous) and $\psi_{\alpha\beta}(x, \dots)$ is a diffeomorphism (homeomorphism) of F for each $x \in U_{\alpha\beta} := U_\alpha \cap U_\beta$. Thus, we may consider the mappings $\psi_{\alpha\beta} : U_{\alpha\beta} \rightarrow G(F)$ with values in the group $G(F)$, $G(F) = \text{Diff}(F)$ is the group of all diffeomorphisms of F or $G(F) = \text{Homeo}(F)$ is the group of all homeomorphisms of F . Mappings $\psi_{\alpha\beta}$ are called the transition functions of the bundle. They satisfy the cocycle conditions: $\psi_{\alpha\beta}(x) \circ \psi_{\beta\gamma}(x) = \psi_{\alpha\gamma}(x)$ for $x \in U_{\alpha\beta\gamma}$ and $\psi_{\alpha\alpha}(x) = Id_F$ for $x \in U_\alpha$. The collection $\{\psi_{\alpha\beta}\}$ is called a cocycle of transition functions.

Given an open cover $\{U_\alpha\}$ of manifold M^n and cocycle of transition functions we may construct a fiber bundle (E, π, M^n, F) .

Principal fiber bundles and vector bundles are the most important cases of fibre bundles.

4°. If $W = M^n \setminus GBH(\varepsilon)$ and $W_0 = \Psi^{-1}(W)$ then $W_0 \subset \text{Int}\delta_0^n$. We consider any piecewise smooth broken line $\gamma = \gamma_0 \cup \gamma_1$ from **2°**. If $\gamma_{01} = \Psi^{-1}(\gamma_0)$ and $\gamma_{02} = \gamma_0 \setminus \gamma_1$ then $\gamma = \gamma_{01} \cup \gamma_{02} \cup \gamma_1$. We define a homeomorphism $\bar{\Psi} : M^n \rightarrow M^n$ by the following conditions:

- a) $\bar{\Psi}|_{W_0} = \Psi|_{W_0}$ i.e. $\bar{\Psi}(\gamma_{01}) = \gamma_0$ and $\bar{\Psi}(W_0) = W$;
- b) $\bar{\Psi}$ maps every segment $\gamma_{02} \cup \gamma_1$ on the segment γ_1 by the length as it was shown above;
- c) $\bar{\Psi}(z) = z$ for every $z \in K^{n-1}$.

It is evident that $\bar{\Psi}$ is a sectionally-smooth homeomorphism.

Let (E, π, M^n, F) be a smooth fibre bundle with a collection fibre charts (U_α, ψ_α) . We can choose such a triangulation, let it be initial one, that $W_0 \subset U_0$. We define $\bar{U}_\alpha = \bar{\Psi}(U_\alpha)$ and $\bar{\psi}_{\alpha\beta}(x) = \psi_{\alpha\beta}(\bar{\Psi}^{-1}(x))$.

The open cover $\{\bar{U}_\alpha\}$ of the manifold M^n and the cocycle $\{\bar{\psi}_{\alpha\beta}\}$ defines a continuous and sectionally-smooth fiber bundle $(\bar{E}, \bar{\pi}, M^n, F)$.

Since $\bar{U}_0 = \bar{\Psi}(U_0) \supset W$ it follows that the fiber bundle $(\bar{E}, \bar{\pi}, M^n, F)$ is trivial over W i.e. $\bar{\psi}_0 : \bar{E}|_W \leftrightarrow W \times F$.

Definition 3. *The fiber bundle $(\bar{E}, \bar{\pi}, M^n, F)$ is called a deformation of the fibre bundle (E, π, M^n, F) towards the $GBH(\varepsilon)$.*

Such characteristics of (E, π, M^n, F) as connections, curvatures *etc* play an important role in the gauge theory, [6].

Problem. *It seems to be interesting to consider good defined deformations of the characteristics above towards the $GBH(\varepsilon)$ i.e. to obtain some similar characteristics of $(\bar{E}, \bar{\pi}, M^n, F)$.*

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